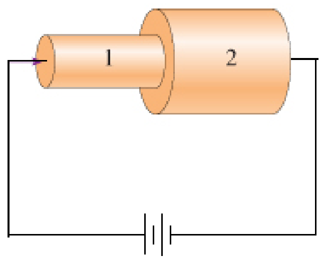
Homework 8 (Solutions): Serkits

Problem 1. Two resistors are hooked up to a battery. They are made of identical material and are of identical length. Resistor 2 is obviously wider than resistor 1. So…(and provide justification).



(a) is R_1 greater than, less than, or equal to R_2?
Since $R = \rho L/A$, and ρL is the same for both, resistor 1 must have the largest R because it has the smallest A. So $\mathbf{R_1 > R_2}$.

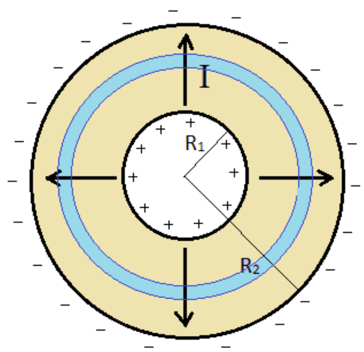
(b) is I_1 greater than, less than or equal to I_2?
$\mathbf{I_1 = I_2}$, because current cannot change within the same wire.

(c) is ΔV_1 greater than, less than, or equal to ΔV_2?
By the power of Grey Skull!….i mean Ohm's law $\mathbf{\Delta V_1 > \Delta V_2}$ because $\Delta V = IR$, and $R_1 > R_2$.

(d) is $\mathbf{E}_1$ greater than, less than, or equal to $\mathbf{E}_2$?
Well $E = \Delta V/\Delta s$, where Δs is the length of the resistor. And since $\Delta V_1 > \Delta V_2$, while $\Delta s_1 = \Delta s_2$, we must have $\mathbf{E_1 > E_2}$. Basically, the electric field must ramp up in a higher resistance material, in order to keep the charges moving at the same rate.

(e) is the electron drift velocity in wire 1 greater than, less than, or equal to the drift velocity in wire 2?
The drift velocity, i.e., 'the velocity', is part of the equation $I = nevA$. Now n and e are the same for both resistors, as is I. So if A goes up, then v must go down, to keep I constant. And so v_2 must be less than v_1. So $\mathbf{v_1 > v_2}$.

Problem 2. A spherical capacitor (R_1 = 3mm, R_2 = 5mm) is discharging radially outward through its dielectric. The dielectric has resistivity of 48μΩ·m. (a) If the current passing radius r = 4mm is 7.8A, what is the electric field strength at this radius? (b) And if the speed of the charges at the outer radius is v_2 = 17μm/s, what is their speed, v_1, at the inner radius? (Maaaybe want to see how I relates to electric field, and also use current conservation).



Ohm's law can only be applied to a region where the field is approximately constant. Let the width of that region be Δs, and cross section area A = 4πr². Then, by Ohm's law:

$$I = \frac{\Delta V}{R} = \frac{E\Delta s}{\frac{\rho \Delta s}{A}} = \frac{EA}{\rho}$$

Therefore,

$$E = \frac{\rho I}{A} = \frac{(48\mu\Omega \cdot \mathrm{m})(7.8A)}{4\pi(4\mathrm{mm})^2} = 1.9\,\mathrm{V/m}$$

And current conservation will give us the speed at the inner radius:

$$I_{inner} = I_{outer}$$
$$nev_{inner}A_{inner} = nev_{outer}A_{outer}$$
$$v_{inner} = v_{outer}\frac{A_{outer}}{A_{inner}} = 17\mu\mathrm{m/s}\frac{4\pi(5\mathrm{mm})^2}{4\pi(3\mathrm{mm})^2} = 47\mu\mathrm{m/s}$$

Problem 3. The electron beam inside a television picture tube is 0.50 mm in diameter and carries a current of 65 μA. This electron beam collides with the inside of the screen. If the electrons are traveling at a speed of v = 3×10^7 m/s, what power is delivered to the screen? (Probably want to do Power = Δ(KE)/Δt).

Thanks for the suggestion, Andrew. No problem bruh.

$$P = \frac{\Delta KE}{\Delta t} = \frac{\Delta N \cdot \frac{1}{2}mv^2}{\Delta t}$$

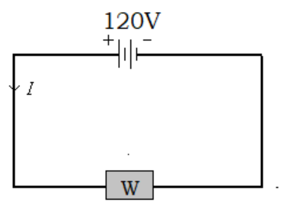
Where ΔN is the number of electrons that hit the screen in time ΔN. ΔN/Δt can be related to the current,

$$I=\frac{\Delta q}{\Delta t}=\frac{e\Delta N}{\Delta t}\rightarrow\frac{\Delta N}{\Delta t}=\frac{1}{e}I$$

So,

$$P=\frac{I}{e}\cdot\frac{1}{2}mv^2=\frac{65\mu\text{A}}{1.6\times10^{-19}\text{C}}\frac{1}{2}(9.11\times10^{-31}\,\text{kg})(3\times10^7\,\text{m/s})^2=1.67\times10^5\,\mu\text{W}=0.167\text{W}$$

Problem 4. Say I have a lightbulb with a tungsten filament (diameter = 0.03mm, length = 10cm, ρ = 5×10^{-8} Ωm, density = 1900 kg/m^3, atomic mass = 183g) hooked up to a 120V battery via copper wire (diameter = 3mm, L = 20cm, ρ = 2×10^{-8} Ωm, density = 9000 kg/m^3, atomic mass = 63g). Note the length of the copper wire is the length of the whole wire, not just one side of it.



(a) What is the current in the wire? In the tungsten filament?
First need the resistances of these guys:

$$R_W=\frac{\rho_W L}{A}=\frac{(50\times10^{-9})(0.10)}{\pi(0.015\times10^{-3})^2}=7.1\Omega$$
$$R_{Cu}=\frac{\rho_{Cu} L}{A}=\frac{(20\times10^{-9})(0.20)}{\pi(1.5\times10^{-3})^2}=5.7\times10^{-4}\Omega$$

So the total resistancee is:

$$R_{total}=7.1\Omega+5.7\times10^{-4}\Omega=7.1\Omega$$

And so the current will be:

$$I=\frac{\Delta V}{R_{total}}=\frac{120\text{V}}{7.1\Omega}=16.9\text{A}$$

(b) What is the power supplied by the battery? What is the power dissipated by the copper wire, by the tungsten filament?
Power supplied by battery: $P=I\Delta V=(16.9)(120)=2\text{kW}$
Power dissipated by copper wire: $P=I^2R=(16.9)^2(5.7\times10^{-4})=0.16\text{W}$
Power dissipated by tungsten filament: $P=I^2R=(16.9)^2(7.1)=2\text{kW}$

(c) What is the electric field inside the copper wire? Inside the tungsten filament?

Electric field is just E = ΔV/Δs. So,

$$E_W = \frac{\Delta V_W}{\Delta s_W} = \frac{IR_W}{\Delta s_W} = \frac{(16.9)(7.1)}{0.10} = 1200\,\mathrm{N/C}$$
$$E_{Cu} = \frac{\Delta V_{Cu}}{\Delta s_{Cu}} = \frac{IR_{Cu}}{\Delta s_{Cu}} = \frac{(16.9)(5.7\times10^{-4})}{0.20} = 0.048\,\mathrm{N/C}$$

(d) What is the electron drift velocity inside the copper wire? Tungsten filament? You can assume each contributes one electron per atom to the current.
We need to get n for each now.

$$n_W = \tilde{n}N_A \frac{\rho_{mass,W}}{m_{molar,W}} = (1)(6.022\times10^{23})\frac{1900\,\mathrm{kg/\,m^3}}{0.183\,\mathrm{kg}} = 6.25\times10^{27}$$
$$n_{Cu} = \tilde{n}N_A \frac{\rho_{mass,Cu}}{m_{molar,Cu}} = (1)(6.022\times10^{23})\frac{9000\,\mathrm{kg/\,m^3}}{0.063\,\mathrm{kg}} = 8.6\times10^{28}$$

And so then from I = nevA, it follows that v = I/neA,

$$v_W = \frac{I}{n_W eA_W} = \frac{16.9}{(6.25\times10^{27})(1.6\times10^{-19})\pi(0.015\times10^{-3})^2} = 24\,\mathrm{m/s}$$
$$v_{Cu} = \frac{I}{n_{Cu} eA_{Cu}} = \frac{16.9}{(8.6\times10^{28})(1.6\times10^{-19})\pi(1.5\times10^{-3})^2} = 1.7\times10^{-4}\,\mathrm{m/s}$$

(e) What are the electrons' mean free collision time inside the copper wire? Tungsten filament?
There are various ways to get this, but one is from $\rho = m/ne^2\tau \rightarrow \tau = m/ne^2\rho$,

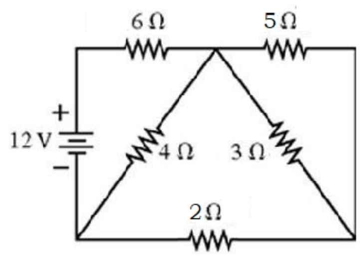
$$\tau_W = \frac{m}{n_W e^2 \rho_W} = \frac{9.11\times10^{-31}}{(6.25\times10^{27})(1.6\times10^{-19})^2(5\times10^{-8})} = 1.14\times10^{-13}\,\mathrm{s}$$
$$\tau_{Cu} = \frac{m}{n_{Cu} e^2 \rho_{Cu}} = \frac{9.11\times10^{-31}}{(8.6\times10^{28})(1.6\times10^{-19})^2(2\times10^{-8})} = 2.1\times10^{-14}\,\mathrm{s}$$

(f) Fiiiinally, say the battery's lifetime is rated to be 10 Amp·hours. How much charge can it transport? How long (Δt) will it be able to light the bulb?
A battery's 'lifetime' is just the amount of charge it can transport. And this is 10 Amp·hours = 10 (C/s)·(3600s) = 3.6×10^4C.

And it will light the bulb for an amount of time,

$$\Delta t = \frac{Q}{I} = \frac{3.6\times10^4\,\mathrm{C}}{16.9\,\mathrm{C/s}} = 2100\,\mathrm{s}$$

Problem 5. Say we hook up a bunch of Christmas tree lights to a 12 battery. Calculate the currents through all resistors, the power dissipated by each resistor, and then rank the resistors/lightbulbs in order of brightness. Last calculate the power supplied by the battery and the total power absorbed by the resistors.

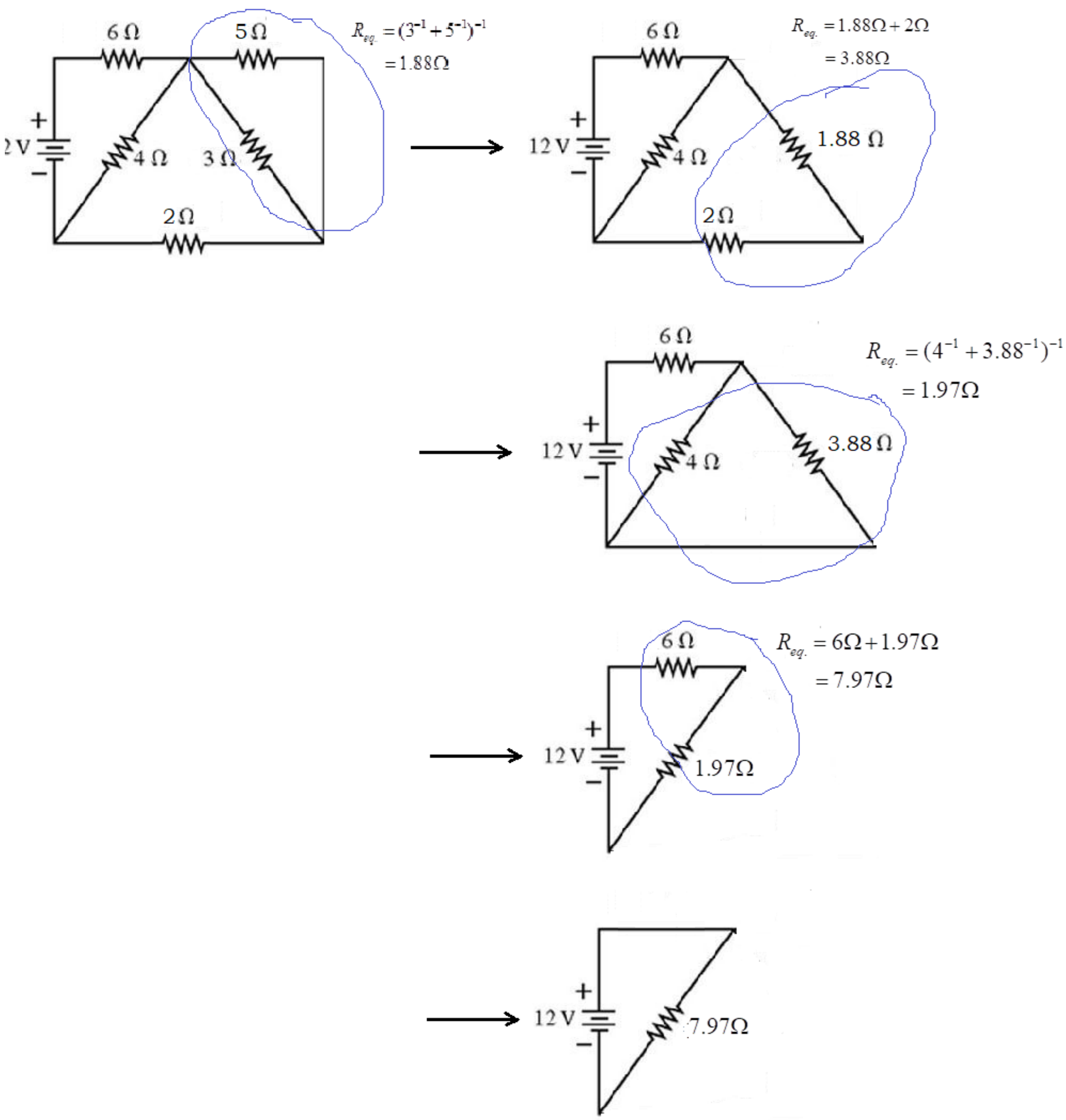


Resistor	Current	Power	Brightness
2Ω			
3Ω			
4Ω			
5Ω			
6Ω			

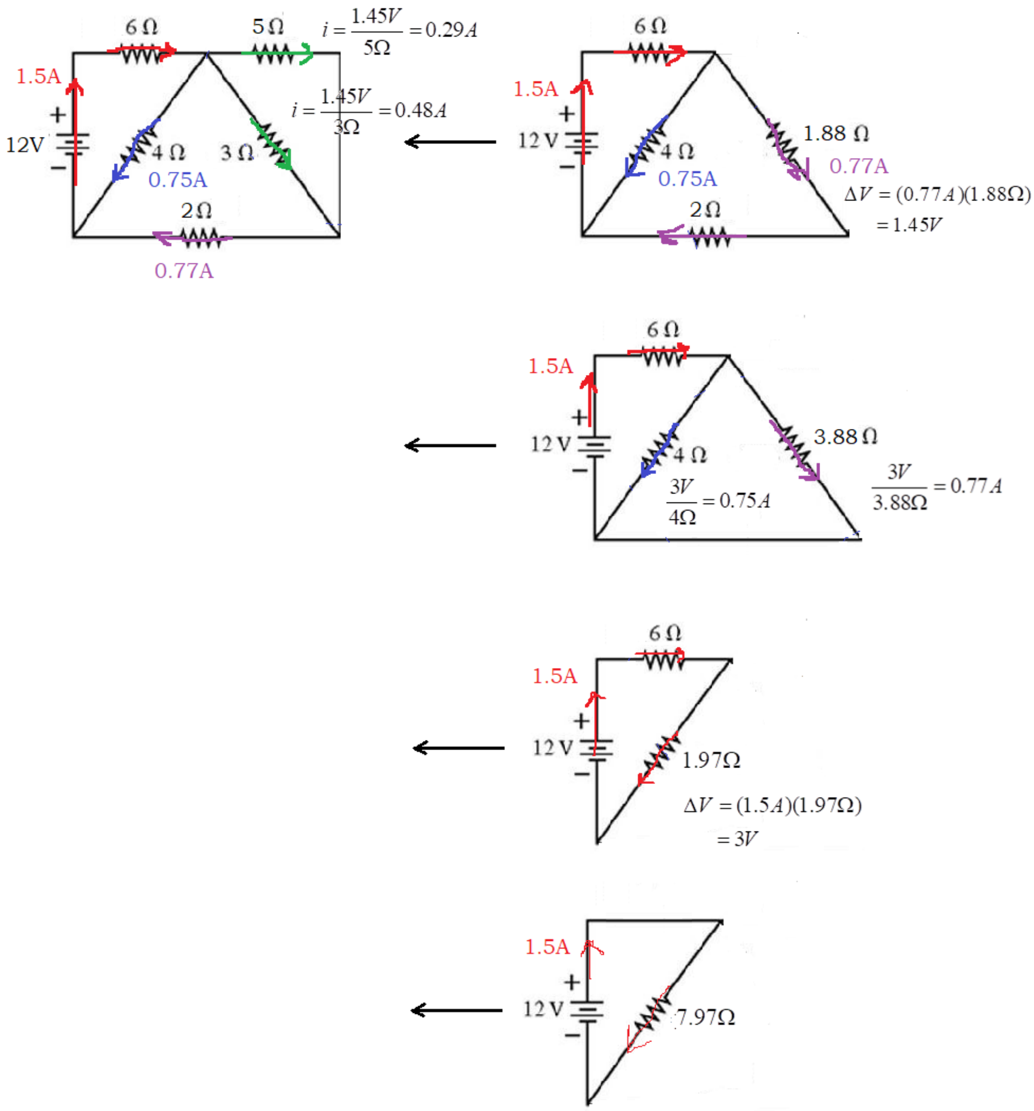
Power supplied	
Power absorbed	

I'll reduce the network to one resistor, step by step.

$$R_{eq.} = 1.88\Omega + 2\Omega$$
$$= 3.88\Omega$$



So the current in the battery is I = 12V/7.97Ω = 1.5A. Then going backwards (bottom to top)



So then working out the power, via I^2R, for each resistor, we get:

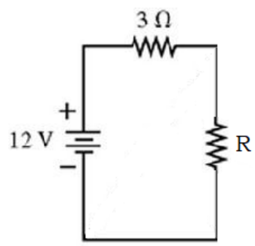
Resistor	Current	Power	Brightness
2Ω	0.77	1.2	3
3Ω	0.48	0.7	4
4Ω	0.75	2.3	2
5Ω	0.29	0.4	5
6Ω	1.5	13.5	1

Power supplied by battery is P = IΔV = (1.5A)(12V) = 18W. Power dissipated by resistors is: 13.5W + 0.4W + 2.3W + 0.7W + 1.2W = 18W, sans a little rounding.

Power supplied	18W
Power absorbed	18W

So power is conserved, as it should be.

Problem 6. Here's an interesting question. Say our 12V battery has an internal resistance of 3Ω (represented by the lovely 3Ω resistor in the diagram). And we want to illuminate our dark existence with the bightest possible light. What resistance lightbulb, R, should we plug into the circuit? Make sure you justify your answer (with calculus).



Power across R is:

$$P = i^2 R = \left(\frac{12}{3+R}\right)^2 R = \frac{144R}{(3+R)^2}$$

And we want to see what R maximizes this P. So differentiate P w/r to R and set derivative equal to zero.

$$\frac{dP}{dR} = 0$$

$$\frac{144(3+R)^2 - 144R \cdot 2(3+R)}{(3+R)^4} = 0$$

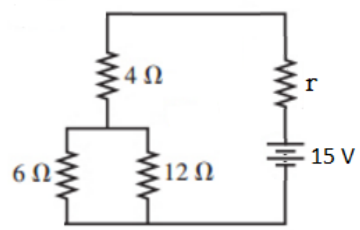
$$144(3+R)^2 - 144R \cdot 2(3+R) = 0$$

$$3 - R = 0$$

$$R = 3\Omega$$

So the max power resistor/lightbulb is the one that matches the internal resistance of the battery. This is known as 'impedance matching'.

Problem 7. Over time, as the terminals on a battery corrode, they get thinner and alloyed with impurities, both of which increase their resistance (recall R = ρL/A). Say you've got a 15V car battery with internal resistance r. And, in my slightly fictional schematic adapted for the purposes of physics, a 6Ω starter motor is connected to it. If the starter motor must receive a potential difference of at least 7V to operate, what maximum value could r be?

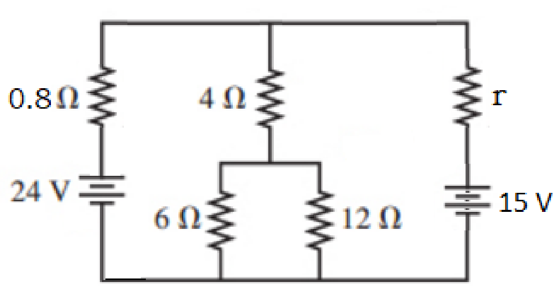


One way…we can work out the potential difference across the 6Ω motor, as a function of the unknown r. Equivalent resistance of the 6Ω and 12Ω resistors is $(6^{-1} + 12^{-1})^{-1} = 4\Omega$. So the circuit's total resistsance is: $R_{eq.} = r + 4 + 4 = r + 8$. Therefore current through the battery is $i = 15/(r+8)$. Now the potential difference across the 6Ω, 12Ω resistors is the potential difference across their equivalent, which is: $\Delta V = iR_{eq.} = 15/(r+8)\cdot 4 = 60/(r+8)$. And this will also be the potential difference across the 6Ω motor, as well as the 12Ω resistor. And we want,

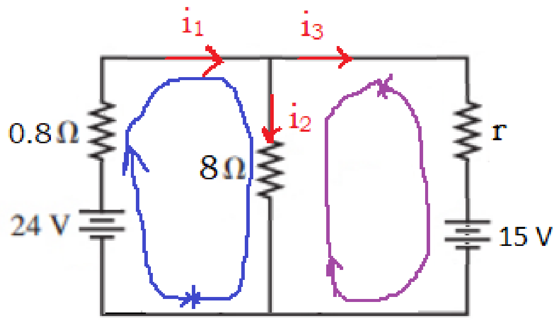
$$\Delta V_6 > 7$$
$$\frac{60}{r+8} > 7$$
$$\frac{60}{7} > r+8$$
$$0.57 > r$$

So r must be less than 0.57Ω.

Problem 8. Let's say the internal resistance, r, of our battery above is 1Ω, which will be insufficient to provide the requisite voltage to the 6Ω starter motor. So we 'jump' the car with another battery: 24V with 0.8Ω internal resistance. (a) What will now be the voltage across the starter motor? Will it start? (b) What will be the total power supplied to the circuit? What will be the total power absorbed?



We can start by combining the middle wire resistors again, and get this circuit:



KCL at the top junction, and KVL around the two loops gives:

$$-i_1 + i_2 + i_3 = 0$$
$$+24 - 0.8i_1 - 8i_2 = 0$$
$$-1i_3 - 15 + 8i_2 = 0 \qquad \text{remember } r = 1\Omega$$

Simplifying a bit:

$$-i_1 + i_2 + i_3 = 0$$
$$0.8i_1 + 8i_2 = 24$$
$$8i_2 - i_3 = 15$$

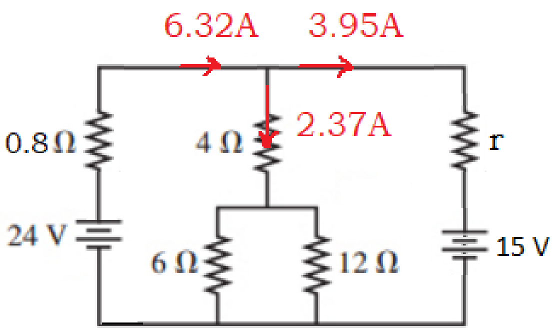
Putting in matrix form,

$$\begin{pmatrix} -1 & 1 & 1 \\ 0.8 & 8 & 0 \\ 0 & 8 & -1 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \\ 15 \end{pmatrix}$$

And solving,

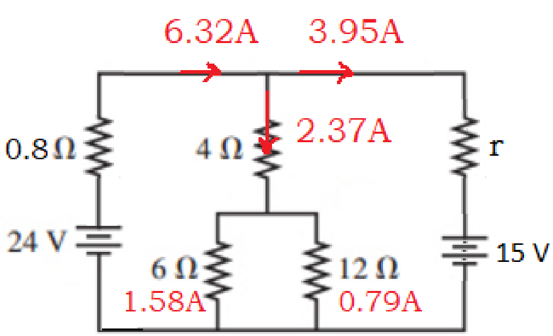
$$\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0.8 & 8 & 0 \\ 0 & 8 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 24 \\ 15 \end{pmatrix}$$
$$= \begin{pmatrix} 6.32 \\ 2.37 \\ 3.95 \end{pmatrix}$$

So now we have:



So the current running down the middle branch is i_2 = 2.37A. And when it crosses the equivalent resistance of the 6Ω,12Ω guys the potential difference will be $\Delta V = i_2 R_{eq.}$ = (2.37A)(4Ω) = 9.48V. This will suffice to start the motor.

And the current in 6Ω motor/resistor will be i = 9.48V/6Ω = 1.58A, and that across the 12Ω resistor will be i = 9.48V/12Ω = 0.79A. So now we have:



Now the power supplied will be the power generated by the 24V battery. Note the 15V battery will be absorbing power rather and delivering it, because the current running through the battery is going from + terminal to – terminal. So,

$$P_{\text{supplied}} = I\Delta V = (6.32A)(24V) = 152\text{W}$$

$$\begin{aligned} P_{absorbed} &= (6.32A)^2(0.8\Omega) + (2.37A)^2(4\Omega) + (1.58A)^2(6\Omega) + (0.79A)^2(12\Omega) \\ &\quad + (3.95A)^2(1\Omega) + (3.95A)(15V) \\ &= 152\text{W} \end{aligned}$$

So they match. Das gud.